LESSON

19 Integration by Substitution

The opposite of the Chain Rule is an integration technique called *substitution*. Using the Chain Rule, for example, the derivative of $8(3x^2 + 7)^5$ is $\frac{d}{dx}(8(3x^2 + 7)^5) = 8 \cdot 5(3x^2 + 7)^4 \cdot 6x = 240x(3x^2 7)^4$. The corresponding antiderivative is thus $\int 240x(3x^2 + 7)^4\,dx = 8(3x^2 + 7)^5 + c$. It is easy to recognize this after seeing the derivative worked out, but how should we know this otherwise?

The mantra of the Chain Rule is "multiply by the derivative of the inside." So the first step to undoing it is to identify what "the inside" must have been. We substitute a new variable u for this and then try to rewrite the whole integral in terms of u.

For example, when confronted by $\int 240x(3x^2 + 7)^4\,dx$, we first notice that this is not an easy integral to solve. If we multiplied out the fourth power, then it would be a polynomial that we know how to evaluate, but that would be quite difficult. Instead, we guess that "the inside" is the stuff inside the parentheses, and substitute $u = 3x^2 + 7$.

To convert the integral entirely over to u, we will need to replace the dx into a du. Because $u = 3x^2 + 7$, we know that $\frac{du}{dx} = 6x$. It is technically wrong to cross-multiply and say $du = 6x\,dx$, but this does result in the correct answer, so we'll go with it. Thus, $dx = \frac{du}{6x}$. The process of substitution works as follows.

Start with the original integral.

$$\int 240x(3x^2 + 7)^4\,dx$$

Substitute $u = 3x^2 + 7$ and $dx = \frac{du}{6x}$.

$$\int 240x(u)^4\frac{du}{6x}$$

Simplify.

$$\int 40u^4\,du$$

Evaluate.

$$8u^5 + c$$

Replace $u = 3x^2 + 7$.

$$8(3x^2 + 7)^5 + c$$

Thus, $\int 240x(3x^2 + 7)^4\,dx = 8(3x^2 + 7)^5 + c$, as we already knew.

In general, try using something inside parentheses with u. If every x doesn't cancel out when replacing dx with du, then try using something else as u. Sometimes, the entire denominator can be used as u. Sometimes, nothing works and a different technique must be tried.

Example

Evaluate $\int x^2\sin(x^3)\,dx$.

Solution

If we use the stuff inside the only set of parentheses, then $u = x^3$, and thus $du = 3x^2\,dx$ and $dx = \frac{du}{3x^2}$.

Start with the original integral.

$$\int x^2\sin(x^3)\,dx$$

Substitute $u = x^3$ and $dx = \frac{du}{3x^2}$.

$$\int x^2\sin(u)\frac{du}{3x^2}$$

Simplify.

$$\int \frac{1}{3}\sin(u)\,du$$

Every x is gone, so we can evaluate.

$$-\frac{1}{3}\cos(u) + c$$

Replace $u = x^3$.

$$-\frac{1}{3}\cos(x^3) + c$$

Thus, $\int x^2\sin(x^3)\,dx = -\frac{1}{3}\cos(x^3) + c$. This can be verified by differentiating $\frac{d}{dx}\left(-\frac{1}{3}\cos(x^3) + c\right) = -\frac{1}{3}(-\sin(x^3)\cdot 3x^2) + 0 = x^2\sin(x^3)$.

If we had been faced with $\int \sin(x^3)\,dx$ in the last example, then substituting $u = x^3$ would have resulted in $\int \sin(u)\frac{du}{3x^2}$. This cannot be evaluated because it is not entirely in terms of u. In fact, this integral is very difficult to solve and requires the advanced technique of replacing $\sin(x^3)$ with an infinitely long polynomial called a *power series*. Many such integrals exist that are difficult to solve, and some have completely baffled every effort to solve them so far. This book will focus on the ones that can be evaluated with basic techniques.

Example

Evaluate $\int \frac{3}{2x+7}dx$.

Solution

Because there are no parentheses, try using the denominator: $u = 2x + 7$. Here, $\frac{du}{dx} = 2$, so $du = 2dx$ and $dx = \frac{du}{2}$.

Start with the original integral.

$$\int \frac{3}{2x+7}dx$$

Substitute $u = 2x + 7$ and $dx = \frac{du}{2}$.

$$\int \frac{3}{u}\frac{du}{2}$$

Simplify.

$$\int \frac{3}{2}\cdot\frac{1}{u}du$$

Every x is gone, so we can evaluate.

$$\frac{3}{2}\ln|u| + c$$

Replace $u = 2x + 7$.

$$\frac{3}{2}\ln|2x+7| + c$$

Thus, $\int \frac{3}{2x+7}dx = \frac{3}{2}\ln|2x+7| + c$.

Basically, the dream is to find a u whose derivative is elsewhere in the integral, so that between the u and the du, every x goes away. This leads to some clever tricks, as will be demonstrated in the following examples.

Example

Evaluate $\int \frac{\ln(x)}{x}dx$.

Solution

Here, we use $u = \ln(x)$. This is not because it is in parentheses but because its derivative $\frac{du}{dx} = \frac{1}{x}$ makes up the rest of the integral. Here, $du = \frac{1}{x}dx$, so $dx = x\,du$.

Start with the original integral.

$$\int \frac{\ln(x)}{x}dx$$

Substitute $u = \ln(x)$ and $dx = x\,du$.

$$\int \frac{u}{x}(x\,du)$$

Simplify.

$$\int u\,du$$

Every x is gone, so we can evaluate.

$$\frac{1}{2}u^2 + c$$

Replace $u = \ln(x)$.

$$\frac{1}{2}(\ln(x))^2 + c$$

Thus, $\int \frac{\ln(x)}{x}\,dx = \frac{1}{2}(\ln(x))^2 + c$.

Example

Evaluate $\int \sin(x)\cos^3(x)\,dx$.

Solution

Here, the trick is to use $u = \cos(x)$ so that $\frac{du}{dx} = -\sin(x)$ and $dx = -\frac{du}{\sin(x)}$.

Start with the original integral.

$$\int \sin(x)\cos^3(x)\,dx$$

Substitute $u = \cos(x)$ and $dx = -\frac{du}{\sin(x)}$.

$$\int \sin(x)\cdot u^3\left(-\frac{du}{\sin(x)}\right)$$

Simplify.

$$\int -u^3\,du$$

Every x is gone, so we can evaluate.

$$-\frac{1}{4}u^4 + c$$

Replace $u = \cos(x)$.

$$-\frac{1}{4}\cos^4(x) + c$$

Thus, $\int \sin(x)\cos^3(x)\,dx = -\frac{1}{4}\cos^4(x) + c$.

To use substitution on a definite integral, it is best to evaluate the indefinite integral first.

Example

Evaluate $\int_1^5 \sqrt{3x+1}\,dx$.

Solution

First, we evaluate $\int \sqrt{3x+1}\,dx$ using $u = 3x + 1$, $du = 3\,dx$, and $dx = \frac{du}{3}$.

Start with the original integral.

$$\int \sqrt{3x+1}\,dx$$

Substitute $u = 3x + 1$ and $dx = \frac{du}{3}$.

$$\int \sqrt{u}\,\frac{du}{3}$$

Simplify.

$$\int \frac{1}{3}u^{\frac{1}{2}}\,du$$

Every x is gone, so we can evaluate.

$$\frac{1}{3}\cdot\frac{2}{3}u^{\frac{3}{2}} + c$$

Replace $u = 3x + 1$.

$$\frac{2}{9}(3x + 1)^{\frac{3}{2}} + c$$

Because $\int \sqrt{3x + 1}\,dx = \frac{2}{9}(3x + 1)^{\frac{3}{2}} + c$, it follows that $\int_1^5 \sqrt{3x + 1}\,dx = \left[\frac{2}{9}(3x + 1)^{\frac{3}{2}}\right]_1^5 = \frac{2}{9}(16)^{\frac{3}{2}} - \frac{2}{9}(4)^{\frac{3}{2}} = \frac{2}{9}(64 - 8) = \frac{112}{9}$.

If the wrong u is chosen, then either some of the variables x will still remain or else the simplified integral will still be hard to solve. If this happens, go back to the beginning and try a different u. Don't forget that many integrals, like those of the previous lesson, don't require substitution at all. Like much of mathematics, integration often requires patience and a knack that is developed with practice.

▶ Practice

Evaluate the following integrals.

1. $\int x^4(x^5 + 1)^7\,dx$

2. $\int (4x + 3)^{10}\,dx$

3. $\int_0^1 x^2(x^3 - 1)^4\,dx$

4. $\int (x^3 - 9x + 4)\,dx$

5. $\int x\sqrt{x^2 - 1}\,dx$

6. $\int_1^4 3\sqrt{x}\,dx$

7. $\int_0^7 \sqrt{3x + 4}\,dx$

8. $\int \frac{9x^2 - 5}{3x^3 - 5x}\,dx$

9. $\int 2x^3\cos(x^4)\,dx$

10. $\int \frac{6x^3 - 1}{\sqrt{3x^4 - 2x + 1}}\,dx$

11. $\int (8x + 5)(4x^2 + 5x - 1)^3\,dx$

12. $\int \frac{x}{(4x^2 + 5)^3}\,dx$

13. $\int \frac{1}{4x + 10}\,dx$

14. $\int \sin(x)\cos(x)\,dx$

15. $\int \sin^2(x)\cos(x)\,dx$

16. $\int \cos(4x)\,dx$

17. $\int 4\cos(x)\,dx$

18. $\int \sin(7x - 2)\,dx$

19. $\int e^x\sin(e^x)\,dx$

20. $\int \frac{(\ln(x))^3}{x}\,dx$

21. $\int \frac{1}{x\ln(x)}\,dx$

22. $\int xe^{(x^2)}\,dx$

23. $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$

24. $\int \frac{e^x}{1 + e^x}\,dx$